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A CYLINDRICAL PROBE IN AN ELECTROHYDRODYNAMIC FLOW, NON-COLLINEAR WITH THE ELECTRIC FIELD VECTOR*

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A solution is obtained for the problem of a cylindrical probe with arbitrary (non-circular), well streamlined profile, situated at zero angle of attack in an incompressible, electrohydrodynamic (EHD) flow, without taking into account the effect of electric space charge in the flow and the resistance of the probe. The general case, in which the velocity of incoming flow is directed at an arbitrary angle to the unperturbed electric field strength, is discussed. All modes of the flow past the probe are studied and the corresponding current-charge characteristics of the probe are obtained. A method of diagnosing the concentrations of various types of charged particles in the flows of multicomponent and multiphase media is given. The results obtained can be used in developing a method of diagnosing various EHD flows in nature and technology, and in the processes of coating streamlined bodies using an electric field.

The current-charge characteristic of a probe in a medium at rest was obtained in /1/ for a cylindrical probe of circular cross-section, and a number of special cases of EHD flow, non-collinear with the direction of the unperturbed electric field past a probe, were studied in /2/. However, the current-charge characteristics of a probe obtained in /2/ referred only to the modes of flow corresponding to EHD flows collinear with the direction of the electric field.

1. We shall consider the plane steady state problem of EHD flow past a cylindrical metal probe. Let the parameter of EHD interaction, characterizing the reaction of the electric field on the motion of the medium, be vanishingly small, and let the probe have a streamlined profile positioned at zero angle at attack in the incoming flow. In this connection we shall regard the velocity field of the medium as given, and approximate it with a velocity distribution corresponding to a potential attached, irrotational flow of an incompressible medium /3/, which is admissible when $\mathrm{Re} \equiv 2Ru^0/\nu \gg 1$, $\mathrm{M} \equiv u^0/c \ll 1$ where R is the characteristic transverse size of the probe, $\mathbf{u}^*, \nu$ is the velocity and kinematic viscosity of the medium and c is the velocity of sound in the medium. Here and henceforth the superscript ^{0} will denote the corresponding parameters of the EHD flow unperturbed by the probe at the point where it is inserted.

When computing the perturbations of the electric field caused by the probe, we shall neglect the effect of the eletric space charge in the flow and electric resistance of the probe material. This is admissible when

$$4\pi R \sum e_j n_j^0 / \varepsilon E^0 \ll 1, \quad \sum e_j b_j n_j^0 / \sigma_p \ll 1 \tag{1.1}$$

where e_j, b_j, n_j is the charge, mobility and density of the j-th type of particle ($j = 1, 2, \ldots, N$) in the EHD flow, E^* is the electric field strength ε is the permittivity of the medium and σ_p is the conductivity of the probe material. We mean by the particles the ions or charged particles of the dispersed phase, small enough to have negligible inertia and for their resistance to satisfy Stokes's law.

The inequalities (1.1) always hold for metal ions of sufficiently small size. In this case the electric-field distribution around the cylindrical probe carrying the charge Q^* (per unit length) can be approximated by the electric-field distribution around a perfectly conducting body of the same shape and carrying the same charge, with permittivity ε, and in an external electric field of strength $\mathbf{E}^0$ /4/.

The assertion formulated above enables us to write at once an explicit expression for the velocity of the particles of the j-th type $\mathbf{v}_j^* = \mathbf{u}^* + b_j \mathbf{E}^*$ in complex form in the $z = x + iy$ plane:

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$$v_j^x - iv_j^y = |b_j| E^0 \frac{dG_j}{d\zeta} \frac{d\zeta^*}{dz}; \quad \zeta = \frac{\zeta^*}{R}, \quad \zeta^* = \xi^* + i\eta^* \tag{1.2}$$

$$G_j(\zeta, \mathrm{Re}_j^E, Q_j \psi_j) = [\mathrm{Re}_j^E + \exp(-i\psi_j)]\zeta + \tag{1.3}$$
$$[\mathrm{Re}_j^E - \exp(i\psi_j)]\frac{1}{\zeta} + 2Q \,\mathrm{Ln}\, \zeta$$

$$\mathrm{Re}_j = \frac{u^0}{|b_j| E^0}, \quad Q_j = \frac{b_j Q^*}{\varepsilon R |b_j| E^0}, \quad \psi_j = \arccos \frac{b_j (\mathbf{E}^0 \mathbf{u}^0)}{|b_j| E^0 u^0} \tag{1.4}$$

Here x, y is a Cartesian system of coordinates in the plane of the flow, chosen so that $\mathbf{u}^0 = (u^0, 0)$ where $u^0 > 0$, ψ_j is the angle between the vectors $\mathbf{u}^0$ and $b_j\mathbf{E}^0$, $\zeta^*(z)$ is the conformal mapping, on the outside of a circle of radius R in the ζ^* plane, of the outside of the probe in the flow plane, satisfying the condition that $d\zeta^*/\partial z = 1$ when $z = \infty$. This condition uniquely defines the above mapping, and this also includes the quantity R, which can therefore by regarded as the characteristic transverse dimension of the probe.

For example, for a cylindrical probe whose cross-section in the z plane represents an ellipse stretched along the x axis, with the semi-axes $a, b \, (a > b)$, we have

$$\zeta^* = {}^1/_2 (\sqrt{z^2 + b^2 - a^2} + z), \ R = {}^1/_2 (a + b)$$

Here we choose the branch of the root for which $\sqrt{z^2 + b^2 - a^2}/z = 1$ when $z = \infty$. In particular, for a plate of width l we have $a = l/2$, $b = 0$, therefore $R = l/4$.

The product $R \, | \, b_j \, | \, E^0 G_j$ represents a complex velocity potential of the charged particles of j-th type in the ζ^* plane, in the case of a potential, attached, irrotational incompressible EHD flow past a perfectly conducting cylinder $| \, \zeta^* \, | = R$ carrying a charge Q, where the flow has an infinitesimal electric space charge and the following values of $\mathbf{u}^*, \mathbf{E}^*$ at $\zeta^* = \infty$ (in complex representation):

$$(u^\xi + iu^\eta)_{\zeta=\infty} = u^0, \ b_j (E^\xi + iE^\eta)_{\zeta=\infty} = | \, b_j \, | \, E^0 \exp(i\psi_j)$$

When computing the perturbations in the charged particle density caused by the probe, we shall neglect their diffusion and volume reactions involving various types of particles. This is admissible, provided that the following conditions hold:

$$\mathrm{Pe}_j^E \equiv \frac{R |b_j| E^0}{D_j} \gg 1, \quad \frac{n_j |b_j| E^0}{R \nu_j^0} \gg 1, \quad \frac{K_j}{|b_j| E^0} \gg 1 \tag{1.5}$$

Here D_j, n_j, ν_j, K_j are the diffusion coefficient, the concentration, local rate of production or annihilation due to volume reactions, and the rate of surface reaction of adhesion and discharge on the probe surface for j-th type particles. The first two conditions of (1.5) always hold in sufficiently strong electric fields, and the last condition holds for metal probes with the property of a high rate of discharge of the charged particles on their surface. As a result, using the equations of balance of various types of charged particles and the fact that the fields $\mathbf{u}$, $\mathbf{E}$ are solenoidal, we obtain the following relation for determining the perturbation in the concentration of j-type particles:

$$(\mathbf{u}^* + b_j\mathbf{E}^*) \, \nabla n_j = 0$$

Hence, it follows that in the formulation of the problem used here, the particle density n_j is constant along the stress lines. For the stream lines of the particles arriving at the probe from infinity, the constant is equal to the value of n_j^0 in steady flow at the point where the probe is introduced, and for the stream lines of the particles originating at the probe the constant is equal to zero since there is no emission of charged particles at the probe, by the conditions of the problem. Using such a distribution of the charged particle density around the probe and taking into account relations (1.2) for the electric current of charged particles of the j-th type J_j^* impinging on the probe (per unit length of the probe), we have the following expression:

$$J_j^* \equiv -e_j n_j^0 \int_{L_j} (v_j \nu) \, dl = -e_j n_j |b_j| E^0 R \,\mathrm{Im}\, [G_j(\zeta_j^+) - G_j(\zeta_j^-)] \tag{1.6}$$

Here the integral is taken over the segment L_j of the probe contour in the z plane, on which the stream lines of the charged, j-th type particles arriving from infinity terminate, $\zeta_j^\pm$ are the points of the circle $| \, \zeta \, | = 1$ in the ζ plane, corresponding to the ends $L_\pm$ of the contour L under conformal mapping $\zeta = \zeta^*(z)/R$. The ends of the contour L are labelled in such a manner, that the motion from L_- to L_+ along L is anticlockwise in the z plane.

We note that conformal mapping does not alter the qualitative pattern of the stream lines of the charged particles. Therefore, in order to determine $\zeta_j^\pm$ it is sufficient to find the ends of the segment of the circle $|\zeta| = 1$ on which the stream lines of charged particles in the ζ plane end after arriving from infinity. The stream lines are images of the stream lines of charged particles in the z plane under conformal mapping $\zeta = \zeta^*(z)/R$. Since the stream function of the charged particles along their stream lines is $\operatorname{Im} G_j(\zeta, \mathrm{Re}_j^E, Q_j, \psi_j) =$ const, it follows that the pattern of the stream lines in the ζ plane, in particular the values $\zeta_j^{\pm}$ and the dimensionless electric current J_j of j-type particles arriving at the probe

$$J_j \equiv \frac{J_j^*}{2\pi \mathrm{Re}_j n_j^0 |b_j| E^0} = -\frac{1}{2\pi} \operatorname{Im}[G_j(\zeta_j^+) - G_j(\zeta_j^-)] \tag{1.7}$$

are fully determined by the dimensionless parameters $\mathrm{Re}_j^E, Q_j, \psi_j$ appearing in the expression (1.3) for $G_j(\zeta)$.

2. Let us now inspect the behaviour of the stream lines of every type of charged particles, and the dependence of the electric currents J_j arriving at the probe on the characteristic parameters $\mathrm{Re}_j^E, Q_j, \psi_j$. Since in the formulation adopted here the deposition of various types of particles at the probe takes place independently, it is sufficient to carry out the investigation for any single j-th type particle. In the rest of this section we shall neglect the index j on the quantities $J_j, \mathrm{Re}_j^E, Q_j, \psi_j$.

Let us first consider the case $\mathrm{Re}^E > 1$ and $0 < Q < 1$. Fig.1 shows the transformation of the qualitative pattern of stream lines in the ζ plane, with the angle ψ between the vectors $\mathbf{u}^0, t\mathbf{E}^0$ decreasing from π to 0. We see that when $\psi = \pi$, there are no stream lines arriving at the probe surface from infinity (Fig.1a). Therefore when $\psi = \pi$, we have $J = 0$. The symmetry is violated when ψ decreases and continuous deformation of the stream line occurs, but their qualitative behaviour remains the same as in Fig.1a, until the separatrices $O_- l_- I_-$ and $B_+ l_+ I_+$ merge at some critical value $\psi = \psi_c$. When $\psi < \psi_c$, the separatrices split again, the qualitative pattern of the stream lines take the form shown in Fig.1b, stream lines arriving at the probe from infinity appear, and we have $J > 0$. When the angle ψ decreases further the current J increases until the upper (at $\psi = \psi_+$) and lower (at $\psi = \psi_-$) shaded zones (see Fig.1b) adjacent to the probe vanish consecutively for values $\psi = \psi_{\pm}$ where $\psi_{\pm} = \arccos(\mp Q)$, after which the current J becomes independent of ψ.

When $\psi < \psi_{\pm}$, the stream lines will come into tangential contact with the circle $|\zeta| = 1$ at the points $\zeta_{\pm}$.

This enables us to obtain, using relations (1.3) and (1.7), exact analytical expressions for $\zeta^{\pm}$ (at $\psi < \psi_{\pm}$) and for J (at $\psi < \psi_-$)

$$\zeta^{\pm} = e^{i\psi}(-Q^* \mp i\sqrt{1 - Q^{*2}}) = \exp[i(\pi + \psi \pm \arccos Q^*)];\ \psi < \psi_{\pm} \tag{2.1}$$

$$J = 2\pi^{-1}(\sqrt{1 - Q^2} - Q \arccos Q);\ 0 \leqslant \psi < \arccos Q \tag{2.2}$$

Since $\operatorname{Im} G = \text{const}$ along the stream lines, it follows that when $\psi > \psi_{\pm}$ we have $\operatorname{Im} G(\zeta^{\pm}) = \operatorname{Im} G(l_{\pm})$ (see Fig.1b) and the singularities will be found from the condition $dG/d\zeta = 0$.

$$l_{\pm} = \frac{-Q}{\mathrm{Re}^E + e^{-i\psi}} \pm \sqrt{\left(\frac{Q}{\mathrm{Re}^E + e^{-i\psi}}\right)^2 + \frac{\mathrm{Re}^E - e^{i\psi}}{\mathrm{Re}^E + e^{-i\psi}}} \tag{2.3}$$

Numerical computations using formula (1.7) and taking into account the above conditions and relations (1.3) and (2.1)-(2.3), lead to the relations $J(\mathrm{Re}^E, Q, \psi)$ shown in Fig.2 and covering the whole domain of variation of the angle ψ.

Let now consider the case $\mathrm{Re}^E < 1, 0 < Q < 1$. The transformation of the qualitative pattern of stream lines of charged particles in the ζ plane which takes place when ψ decreases from π to 0, is shown in Fig.3 ($Q < Q_0, Q_0 = \sqrt{1 - (Re^E)^2}$) and Fig.4 ($Q > Q_0$). When $\psi = \pi$ and $Q < Q_0$, we have the following expression for the current based on relations (1.7), (2.3) and $\operatorname{Im} G(\zeta^{\pm}) = \operatorname{Im} G(l_{\pm})$ (see Fig.3a):

$$J = 2\pi^{-1}[\sqrt{1 - (\mathrm{Re}^E)^2 - Q^2} - Q \arccos(Q/Q_0)];\ \psi = \pi,\ \mathrm{Re}^E < 1 \tag{2.4}$$

The current J changes as ψ decreases until $\psi = \psi_{\pm}$, whereupon the upper (at $\psi = \psi_+$) and lower (at $\psi = \psi_-$) shaded zones in Fig.3b vanish one after the other. After this (i.e. when $0 < \psi < \psi_-$) the current J remains constant and equal to (2.1). If on the other hand $\psi = \pi$ and $Q > Q_0$, we have no stream lines arriving from infinity at the probe surface (see Fig.4a), and therefore $J = 0$. This situation is maintained as the stream lines deform, corresponding to the reduction in ψ, to some critical value ψ_{cc}. When $\psi < \psi_{cc}$, the stream lines mentioned above appear, and $J > 0$. Figs.4b and 4c show the change in the qualitative pattern of the stream lines when the angle ψ passes through the value ψ_{cc}. When there is a further reduction in ψ the current J changes, until when $\psi = \psi_{\pm}$ the upper and lower shaded zones in Fig.4c disappear, one after the other. After this (i.e. when $0 < \psi < \psi_{-}$), the current J remains constant and equal to (2.1).

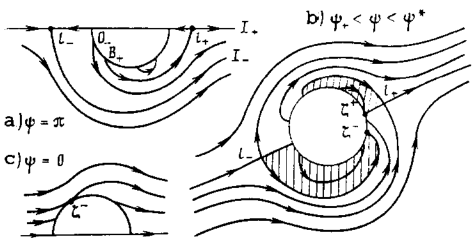

Fig.1

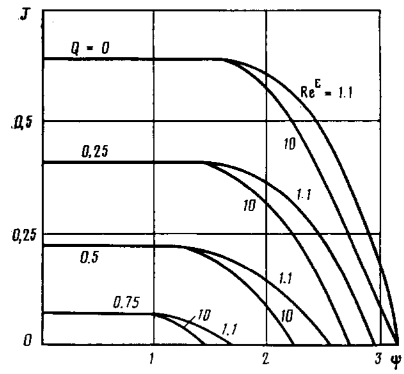

Fig.2

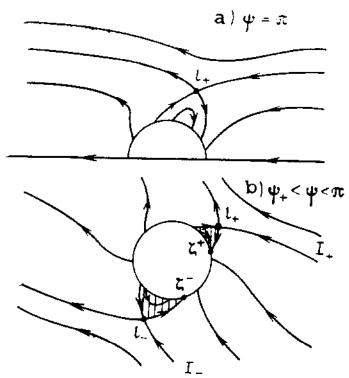

Fig.3

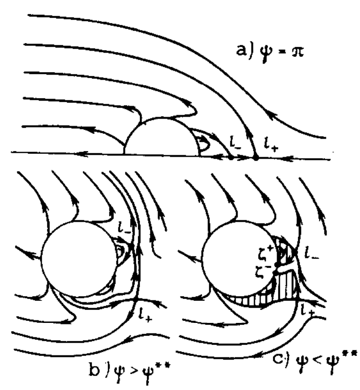

Fig.4

Numerical computations of the current J (over the whole range $0 < Q < 1$, $0 < \psi < \pi$) taking into account relations (1.7), (2.1)-(2.4) and the fact that when $\psi > \psi_{\pm}$ we have $\operatorname{Im}(\zeta^{\pm}) = \operatorname{Im} G(l_{\pm})$ (see Fig.3b and 4c), yield when $\mathrm{Re}^E < 1$ the relations $J^*(\mathrm{Re}^E, Q, \psi)$ shown in Fig.5 and 6.

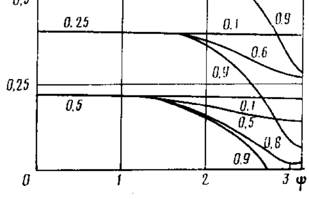

Fig.5

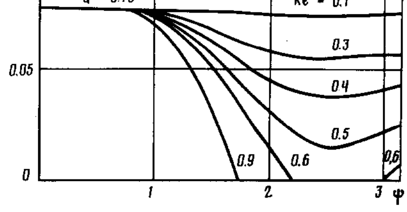

Fig.6

We note that the relation $J(\psi)$ is not monotonic when $\mathrm{Re}^E < 1$ and $Q < Q_0$. This is due to the fact that, as calculations show, when the angle ψ decreases, the channel $I_+ l_+ \zeta^+ \zeta^- l_- I_-$ (Fig.3b), along which the particles are deposited on the probe, at first becomes narrower, but later widens again. In particular, when the values of Re^E and Q are sufficiently close to unity and Q_0, respectively, the channel may become completely overlapped at some values of ψ and the current J may vanish as a result (see the relation shown in Fig.6 for $\mathrm{Re}^E = 0.6$, $Q = 0.75$).

Finally we can show that for $Q > 1$ (and arbitrary Re^E, ψ) no stream lines of charged particles arrive at the probe from infinity, and hence $J = 0$.

The study of the qualitative pattern of the stream lines of charged particles and the calculation of their electric current J arriving at the probe can be carried out in exactly the same manner for $Q < 0$. We find that the stream line patterns are obtained, for any value of $Q < 0$, from the stream line patterns given above for the opposite value of $-Q > 0$ by rotating them by 180° and reversing the direction of motion. This implies that the function $J(\mathrm{Re}^E, Q, \psi)$ $(Q \lessgtr 0)$ satisfies the relation

$$J(\mathrm{Re}^E, Q, \psi) - J(\mathrm{Re}^E, -Q, \psi) = -2Q$$

from which we can obtain the relation for $J(\mathrm{Re}^E, Q, \psi)$ and $Q < 0$ in terms of the relation $J(\mathrm{Re}^E, Q, \psi)$ when $Q > 0$ investigated above. We also find that formulas (2.2) and (2.4) remain valid over the whole range $-1 < Q < 1$, and at $|Q| > 1$ we have $J = |Q| - Q$ for any Re^E, ψ.

3. Let us now consider the application of the above theory of the diagnostics of the concentrations of various type charged particles in EHD flows. Using the results obtained we can write the current-charge characteristics of the probe as follows:

$$J^*(Q^*, \mathbf{E}^0, \mathbf{u}^0) = 2\pi R \sum_{j=1}^{N} e_j n_j^0 |b_j| E^0 J_j(Q_j, \mathrm{Re}_j^E, \psi_j) \tag{3.1}$$

By measuring N values $J_k^*(k = 1, \ldots, N)$ of the electric current arriving at the probe for N different sets of parameters Q_k^*, $\mathbf{E}_*^0$ (the magnitude and direction of $\mathbf{E}^0$ can be varied by applying a sufficiently strong external field), we obtain from relation (3.1)

$$J_k^* = \sum_{j=1}^{N} c_{kj} n_j^0, \quad c_{kj} = 2\pi e_j R |b_j| E^0 \times$$
$$\times J_j\left(\frac{b_j Q_k}{\varepsilon R |b_j| E_k^0}, \frac{u^0}{|b_j| E_k^0}, \arccos \frac{b_j (E_k^0 u^0)}{|b_j| E_k^0 u^0}\right) \tag{3.2}$$

The dependence of the function J_j on its arguments was determined above. The system of Eqs.(3.2), linear in n_j^0, enables us to determine the values of the concentrations of all N types of charged particles in the EHD flow over the measured values of J_k^*, Q_k^*, E_k^* $(k = 1, \ldots, N)$, provided that the properties of separate particles of each type (i.e. e_j, b_j), are known, and $\det \| c_k b_j \| \neq 0$. In order to satisfy the last condition in the general case, we must make the measurements at supercritical angles ψ_j, i.e. when $\psi_j > \psi_j^-$. Indeed, when $\psi_j < \psi_j^-$, the functions J_j are given by the relation (2.2) and in case of, for example, unipolarly charged EHD flows they are, in general, independent of j, and $\det \| c_{kj} \| = 0$.

Using the probe for EHD flows for values of the electric field strength which are not collinear with the velocity of the flow, we can, using the measurements of the current-charge characteristics, establish the concentrations of all types of charged particles.

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